COMBINATORICA 3 (2) (1983) 147—152

AN UPPER BOUND FOR THE CARDINALITY OF AN s-DISTANCE SUBSET IN REAL EUCLIDEAN SPACE, II

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Received 14 June 1982

It is shown that if X is an s-distance subset in $\mathbf{R}^d$, then $|X| \leqq \binom{d+s}{s}$.

0. Introduction

A subset X in a metric space M is called an s-distance subset in M if there are s distinct positive distances $\alpha_1, \alpha_2, \ldots, \alpha_s$ and all the α_i are realized.

Larman—Rogers—Seidel [5] proved that $|X| \leqq (d+1)(d+4)/2$ for a 2-distance subset in $\mathbf{R}^d$. Subsequently, Bannai—Bannai [1] proved that $|X| < \binom{d+s}{s} + \binom{d+s-1}{s-1}$ for an s-distance subset in $\mathbf{R}^d$. (That is, $|X| < (d+1)(d+4)/2$ for $s=2$). Then Blokhuis [2] has shown that $|X| \leqq (d+1)(d+2)/2$ for a 2-distance subset in $\mathbf{R}^d$. In the present paper we will generalize the result of Blokhuis [2] for all $s \geqq 2$. Namely, we prove:

Theorem 1. *If X is an s-distance subset in $\mathbf{R}^d$, then we have*

$$|X| \leqq \binom{d+s}{s}.$$

Our basic idea of the proof of Theorem 1 is the same as that of Blokhuis [2]. However, in order to prove the result for all $s \geqq 2$, we had to overcome certain technical complications. Theorem 2 and Theorem 3 which may be of independent interest, serve this purpose. A classical formula of Hobson on spherical harmonic plays an important role in the proof of Theorem 2.

We acknowledge that Theorem 1 was also proved by Blokhuis [3], independently.

[1] Supported in part by NSF grant MCS7903128 A01.
[2] Supported in part by NSF grant MCS.
AMS subject classification (1980): 05 B 99; 51 M 99.

2. Theorem 2

In this section the following theorem, which may be of independent interest, will be proved:

Theorem 2. *Let* $x_1, \ldots, x_d$ *be independent variables, and let us write* $\partial_i = \partial/\partial x_i$. *Let* $0 \leqq l \leqq s+1$. *Then we have:*

$$\text{the space spanned by } \{\partial_1^{b_1}\partial_2^{b_2}\ldots\partial_d^{b_d}(x_1^2+x_2^2+\ldots+x_d^2)^s:\ b_1+\ldots+b_d = 2s-l+1\}$$
$$= \text{the space spanned by } \{x_1^{a_1}x_2^{a_2}\ldots x_d^{a_d}:\ a_1+\ldots+a_d = l-1\}.$$

Remark. Theorem 2 is also stated in the following way. There are $\binom{2s-l+d}{d-1}$ partial differential operators $\partial_1^{b_1}\partial_2^{b_2}\ldots\partial_d^{b_d}$ with $b_1+\ldots+b_d = 2s-l+1$, and there are $\binom{l-2+d}{d-1}$ monomials $x_1^{a_1}x_2^{a_2}\ldots x_d^{a_d}$ with $a_1+\ldots+a_d = l-1$.

Let M be the $\binom{2s-l+d}{d-1} \times \binom{l-2+d}{d-1}$ matrix whose entries (of each row) are the coefficients of each $\partial_1^{b_1}\ldots\partial_d^{b_d}(x_1^2+\ldots+x_d^2)^s$ with respect to $x_1^{a_1}\ldots x_d^{a_d}$. Then M has maximal rank. That is,

(1) $$\text{rank of } M = \binom{l-2+d}{d-1}, \quad \text{if } l = 0, 1, \ldots, s+1, \quad \text{and}$$

(2) $$\text{rank of } M = \binom{2s-l+d}{d-1}, \quad \text{if } l = s+1, \ldots, 2s.$$

In order to prove Theorem 2 we first quote Hobson's formula:

Proposition 2.1. (Hobson) *Let* $P_t(\partial_1, \ldots, \partial_d)$ *be homogeneous polynomial of degree* t *and let* $F(x_1^2+\ldots+x_d^2)$ *be a function of* $r^2 = x_1^2+\ldots+x_d^2$. *Then*

(2.1) $$P_t(\partial_1, \ldots, \partial_d)[F(x_1^2+\ldots+x_d^2)] = \left[\sum_{k=0}^{t} \frac{2^{t-2k}}{k!} \cdot \frac{d^{t-k}}{d(r^2)^{t-k}} \cdot F \cdot \Delta^k\right] P_t(x_1, \ldots, x_d),$$

where Δ *is the Laplacian.*

Proof. See Hobson [4, page 126, Eq. (6)]. Also, Hobson's formula is easily proved by induction on the degree of t in $P_t(\partial_1, \ldots, \partial_d)$. We may assume without loss of generality that P_t is a monomial of degree t, then apply ∂_1 to get

$$[\partial_1 P_t(\partial_1, \ldots, \partial_d)]\, F(r^2) = \sum_k \frac{2^{t-2k}}{k!} \{F^{t-k}(r^2)[\Delta^k \partial_1 P_t] + 2x_1 F^{(t-k+1)}(r^2)\Delta^k P_t$$
$$= \sum_k \frac{2^{t-2k+1}}{k!} F^{(t-k+1)}(r^2)[x_1 \Delta^k P_t + 2k\, \Delta^{k-1}\partial_1 P_t]$$

But $x_1\Delta^k P_t + 2k\Delta^{k-1} = \Delta^k(x_1 P_t)$ so we are done by induction. ∎

Proof of Theorem 2. We need only a special case of Proposition 2.1 to prove Theorem 2. If $H_t(x_1, \ldots, x_d)$ is a homogeneous harmonic polynomial of degree t then

(2.2) $$(H_{t-2j}(\partial_1, \ldots, \partial_d)\Delta^j)(x_1^2+\ldots+x_d^2)^s = M(s, d, t, j)\, r^{2(s-t+j)} H_{t-2j}(x_1, \ldots, x_d),$$

where $M(s, d, t, j)>0$ for $t-2j\geqq 0$ and $t-s\leqq j$, and $M(s, d, t, j)=0$ otherwise. In fact, the relation

$$(2.3)\qquad \Delta\big(H_{t-2j}(x_1, \dots, x_d)r^{2j}\big) = 2j(d+2t-2j-2)H_{t-2j}(x_1, \dots, x_d)r^{2j-2}$$

coupled with Proposition 2.1 gives the following formula:

$$(2.4)\qquad M(s, d, t, j) = \sum_{k=\max(t-s,0)}^{\min([t/2],j)} \frac{2^{t-2k}}{k!} s(s-1)\dots(s-t+k-1)\lambda(t,j)\dots$$
$$\dots\lambda(t-2k+2, j-k+1),$$

where $\lambda(t,j)=2j(d+2t-2j-2)>0$. (In fact, it is easy to evaluate the sum in (2.4) but we do not need this result.)

We find explicitly which polynomials $P_t(\partial_1, \dots, \partial_d)$ annihilate $(x_1^2+\dots+x_d^2)^s$ for Theorem 2. This gives us the rank of the matrix M. Let $P_t(\delta_1, \dots, \delta_d)$ be written uniquely as

$$(2.5)\qquad P_t(\partial_1, \dots, \partial_d) = \sum_{j=0}^{[t/2]} H_{t-2j}(\partial_1, \dots, \partial_d)\Delta^j.$$

So (2.2) implies that $P_t(\delta_1, \dots, \delta_d)(x_1^2+\dots+x_d^2)^s=0$ if and only if

$$(2.6)\qquad P_t(\partial_1, \dots, \partial_d) = \sum_{j=0}^{t-s-1} H_{t-2j}(\partial_1, \dots, \partial_d)\Delta^j.$$

Thus, for $t\leqq s$, $P_t\equiv 0$ and M has rank $\binom{t+d-1}{d-1}=\binom{2s-l+d}{d-1}$ if $l=2s-t+1$ $(=s+1), \dots, 2s+1$. For $t>s$, M has rank

$$(2.7)\qquad \dim \operatorname{Hom}(t)-\sum_{j=0}^{t-s-1} \dim \operatorname{Harm}(t-2j) = \dim \operatorname{Harm}(2s-t).$$

Cleraly, here the rank is

$$\binom{2s-t+d-1}{d-1} = \binom{l+d-2}{d-1} \quad \text{if} \quad l = 2s-t+1$$

This completes the proof of Theorem 2. ∎

3. Theorem 3

In this section we prove the following theorem which may also be of independent interest.

Theorem 3. *For* $i=1,2, \dots, N$ *let* $m_i\in\mathbf{R}$ *and* $y^{(i)}=(y_1^{(i)}, \dots, y_d^{(i)})\in\mathbf{R}^d$. *For fixed integers* $0\leqq l-1\leqq s$ suppose

$$\sum_{i=1}^{N} m_i\|x-y^{(i)}\|^{2s}$$

is a polynomial in $x=(x_1, \ldots, x_d)$ *of degree* $\leqq 2s-l$. *Then*

$$\sum_{i=1}^{N} m_i (y_1^{(i)})^{a_1} \ldots (y_d^{(i)})^{a_d} = 0$$

for any non-negative integers $a_1, \ldots, a_d$ *such that* $0 \leqq a_1+\ldots+a_d \leqq l-1$.

Proof. We have that $\sum_{i=1}^{N} m_i \|x-y^{(i)}\|^{2s}$ is a polynomial of degree $\leqq 2s-l$ in $x_1, \ldots, x_d$ if and only if

$$\partial_1^{b_1}\partial_2^{b_2} \ldots \partial_d^{b_d}\left(\sum_{i=1}^{N} m_i \|x-y^{(i)}\|^{2s}\right) = 0 \tag{3.1}$$

for all $b_1, b_2, \ldots, b_d$ with $2s-l+1 \leqq b_1+\ldots+b_d \leqq 2s$. By Theorem 2 we have

$$(x_1-y_1^{(i)})^{a_1} \ldots (x_d-y_d^{(i)})^{a_d}$$
$$= \sum_{b_1+\ldots+b_d=2s-l+1} C_{b_1\ldots b_d}^{a_1\ldots a_d} \partial_1^{b_1} \ldots \partial_d^{b_d}[(x_1-y_1^{(i)})^2+\ldots+(x_d-y_d^{(i)})^2]^s$$

for some real numbers $C_{b_1\ldots b_d}^{a_1\ldots a_d}$ which do not depend on i.

So, as polynomials in $x_1, \ldots, x_d$, we have

$$\sum_{i=1}^{N} m_i (x_1-y_1^{(i)})^{a_1} \ldots (x_d-y_d^{(i)})^{a_d}$$

$$= \sum_{i=1}^{N} m_i \sum_{b} C_{b_1\ldots b_d}^{a_1\ldots a_d} \partial_1^{b_1} \ldots \partial_1^{b_d}((x_1-y_1^{(i)})^2+\ldots+(x_d-y_d^{(i)})^2)^s$$

$$= \sum_{b} C_{b_1\ldots b_d}^{a_1\ldots a_d} \sum_{i=1}^{N} m_i \partial_1^{b_1} \ldots \partial_d^{b_d}((x_1-y_1^{(i)})+\ldots+(x_d-y_d^{(i)})^2)^2 = 0. \quad \text{(By (3.1).)}$$

By putting $x_1=x_2=\ldots=x_d=0$, we get the desired result. ∎

4. Completion of Proof of Theorem 1

Let X be an s-distance subset in $\mathbf{R}^d$ with s nonzero distances $\alpha_1, \ldots, \alpha_s$. Let us set

$$F_y(x) = \prod_{i=1}^{s} (\|y-x\|^2-\alpha_i^2) / \prod_{i=1}^{s} \alpha_i^2. \tag{4.1}$$

In order to prove Theorem 1, we have only to show that the functions

$$F_y(x), (y \in X), \quad \text{and} \tag{4.2}$$
$$\{x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_d^{\lambda_d} \colon 0 \leqq \lambda_1+\lambda_2+\ldots+\lambda_d \leqq s-1\}$$

are linearly independent functions on $\mathbf{R}^d$, because it is shown in [1] that the space W_s spanned by these functions is of dimension at most $\binom{d+s}{s}+\binom{d+s-1}{s-1}$ and because the space spanned by $\{x_1^{\lambda_1}x_2^{\lambda_2}\dots x_d^{\lambda_d}: 0\leqq\lambda_1+\dots+\lambda_d\leqq s-1\}$ is of dimension $\binom{d+s-1}{s-1}$. Suppose that

$$\sum_{\in X} C_y F_y(x)+\sum_{0<\lambda_1+\dots+\lambda_d<s-1} C_{\lambda_1,\lambda_2,\dots,\lambda_d}\cdot x_1^{\lambda_1}\cdots x_d^{\lambda_d}=0, \tag{4.3}$$

where $C_y(y\in X)$ and the $C_{\lambda_1,\dots,\lambda_d}$ are real numbers. We want to show that these are all 0. For this purpose, it is enough to show that

$$\sum_{y\in X} C_y y_1^{\lambda_1}\dots y_d^{\lambda_d}=0 \quad\text{for}\quad 0\leqq\lambda_1+\lambda_2+\dots+\lambda_d\leqq s-1. \tag{4.4}$$

If we choose $x=u\in X$ in (4.3) we get $(-1)^s C_u+\sum_{\lambda} C_\lambda u^\lambda=0$. Multiplying this by C_u and summing over u yields

$$(-1)^s\cdot\sum_{y\in X} C_y^2+\sum_{0<\lambda_1+\dots+\lambda_d\leqq s-1} C_{\lambda_1,\lambda_2,\dots,\lambda_d}\cdot\sum_{y\in X} C_y y_1^{\lambda_1}y_2^{\lambda_2}\dots y_d^{\lambda_d}=0. \tag{4.5}$$

Then (4.4) implies that

$$\sum_{y\in X} C_y^2=0,\quad\text{and so} \tag{4.6}$$

$$C_y=0\quad\text{for all}\quad y\in X. \tag{4.7}$$

Finally, (4.3) now implies $C_{\lambda_1,\dots,\lambda_d}=0$.

Now, we want to prove (4.4) by induction on $\lambda_1+\dots+\lambda_d$. Comparing the coefficients of x^{2s} in (4.3), we have

$$\sum_{y\in X} C_y=0. \tag{4.8}$$

So, we assume that

$$\sum_{y\in X} C_y y_1^{\lambda_1}\dots y_d^{\lambda_d}=0\quad\text{for all}\quad \lambda_1,\dots,\lambda_d\quad\text{with}\quad \lambda_1+\dots+\lambda_d\leqq l-2, \tag{4.9}$$

and we prove that

(4.10)

$$\sum_{y\in X} C_y y_1^{\lambda_1}\dots y_d^{\lambda_d}=0\quad\text{for all}\quad\lambda_1,\dots,\lambda_d\quad\text{with}\quad\lambda_1+\dots+\lambda_d\leqq l-2\quad\text{if}\quad l\leqq s.$$

Next, we equate coefficients of $x^{2s-(l-1)}$ in (4.3). For $0\leqq l\leqq s$, the second term has no such terms. So the coefficient of $x^{2s-(l-1)}$ in $\sum_{y\in X} C_y F_y(x)$ is zero. We compute this coefficient in another way by expanding (4.1) to find

$$\sum_{y\in X} C_y F_y(x)=\sum_{y\in X} C_y\sum_{t=0}^{s} A_t\|x-y\|^{2(s-t)}=\sum_{t=0}^{s} A_t\sum_{y\in X} C_y\|x-y\|^{2(s-t)} \tag{4.11}$$

for some real numbers $0\neq A_0, A_1, \dots, A_s$. Clearly $\sum_{y\in X} C_y\|x-y\|^{2(s-t)}$ is a homogeneous polynomial in x and y of degree $2(s-t)$. By our assumption (4.9), the y terms of degree 0 to $l-2$ vanish. So, as a polynomial in x, $\sum_{y\in X} C_y\|x-y\|^{2(s-t)}$ has degree $2(s-t)-(l-1)$. Thus the only term in (4.11) which allows degree $2s-(l-1)$ terms in x is $t=0$, and $\sum_{y\in X} C_y\|x-y\|^{2s}$ is a polynomial in x of degree $\leqq 2s-(l-1)$. However, there are no terms in $\sum_{y\in X} C_y F_y(x)$ with x degree $2s-(l-1)$, so $\sum_{y\in X} C_y\|x-y\|^{2s}$ has degree $\leqq 2s-(l-1)-1$. So Theorem 3 (with $N=|X|$, $m_i=C_y$) implies (4.10). Thus, by induction we have shown (4.4). This completes the proof of Theorem 1. ∎

Remark. It would be interesting to know whether there are s-distance subsets in $\mathbf{R}^d$ which attain the equality in Theorem 1. We do not know any such examples with $s\geqq 2$ at present.

References

[1] E. BANNAI and E. BANNAI, An upper bound for the cardinality of an s-distance subset in real Euclidean space, *Combinatorica* **1** (1981), 99—102.

[2] A. BLOKHUIS, A new upper bound for the cardinality of 2-distance sets in Euclidean space, *Eindhoven Univ. of Technology, Memorandum* 1981—**04**, Feb. 1981.

[3] A. BLOKHUIS, An upper bound for the cardinality of s-distance sets in E^d and H^d, *Eindhoven Univ. Techn. Memorandum* 1982—**68**, May 1982. (This is also part of his Ph. D. Thesis entitled *"Few distance sets"*, 1983.)

[4] E. W. HOBSON, *The Theory of Spherical and Ellipsoidal Harmonics*, Cambridge, 1931.

[5] D. G. LARMAN, C. A. ROGERS and J. J. SEIDEL, On two-distance sets in Euclidean space, *Bull. London Math. Soc.* **9** (1977), 261—267.

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